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An Inverse Boundary Value Problem for Maxwell's Equations

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0. Introduction and Preliminaries

In this paper we consider an inverse boundary value problem for Maxwell's equations. This problem was proposed in [S-I-C] and it is the analog, in this context, of the inverse conductivity problem, which has received a great deal of attention in recent years (see for instance the survey paper [S-U I] and the references given there).

The problem we shall consider is, roughly speaking, whether knowledge of the energy needed to maintain a given voltage on the surface of a conductor is enough to determine its electric permittivity, magnetic permeability, and electrical conductivity. We restrict our attention to the case in which the time variation of the electromagnetic field is neglected. We now state more precisely the mathematical problem.

Let $\Omega \subset \mathbb{R}^3$ be a bounded open set with smooth boundary. Let ε_*, μ_* be positive constants, and let $\sigma_* \geqq 0$ and $\omega \in \mathbb{R} - 0$. We assume that $\varepsilon(x)$, $\mu(x) > 0$ in $\bar{\Omega}$, $\sigma(x) \geqq 0$ in $\bar{\Omega}$, and $\varepsilon(x) - \varepsilon_*$, $\mu(x) - \mu_*$, $\sigma(x) - \sigma_* \in C_0^2(\Omega)$. Maxwell's equations for the time-harmonic electromagnetic field are

$$(0.1)\qquad \begin{aligned} \operatorname{curl} e &= \alpha h \quad \text{in } \Omega, \quad \alpha = i\omega\mu, \\ \operatorname{curl} h &= \beta e \quad \text{in } \Omega, \quad \beta = -i\omega\varepsilon + \sigma. \end{aligned}$$

Physically, (e, h) is the time-harmonic electromagnetic field, ω is its frequency, ε denotes the electric permittivity of the conductor Ω, μ denotes the magnetic permeability of Ω, and σ denotes its conductivity. As noted in the discussion in [S-I-C], the total energy through the boundary $\partial\Omega$ is

$$\Phi = \operatorname{Re} \int_{\partial\Omega} \nu \cdot (e \wedge \bar{h})\, dS = \operatorname{Re} \int_{\partial\Omega} (\nu \wedge e) \cdot \bar{h}\, dS,$$

where ν denotes the unit outer normal to $\partial\Omega$ and dS denotes surface measure. This motivated SOMERSALO, ISAACSON & CHENEY to define the boundary map, analogous to the Dirichlet-to-Neumann map, as

$$(0.2)\qquad \Lambda_{\alpha,\beta} : \nu \wedge e|_{\partial\Omega} \to \nu \wedge h|_{\partial\Omega},$$

where e, h satisfy (0.1). In [S-I-C] it was proved that this map is well defined for all but at most a discrete set of frequencies ω_j. From now on we shall assume that ω as in (0.1) is not one of these exceptional frequencies.

The inverse problem we shall consider here is whether knowledge of $\Lambda_{\alpha,\beta}$ determines α and β uniquely in Ω, i.e., whether the map

$$(\alpha, \beta) \xrightarrow{\Lambda} \Lambda_{\alpha,\beta} \tag{0.3}$$

is injective. In [S-I-C] it was proved that the formal linearization of Λ is injective at a constant background. Since the range of Λ is not closed, one cannot deduce local injectivity of Λ by some variant of the implicit function theorem. In this paper we prove local injectivity of Λ near a constant pair (α_*, β_*).

Theorem 0.4. *Let* $\alpha_* = i\omega\mu_*$, $\beta_* = -i\omega\varepsilon_* + \sigma_*$. *Let* ε_j, $\mu_j > 0$ *in* $\bar{\Omega}$, $\sigma \geqq 0$ *in* $\bar{\Omega}$ *and* $\varepsilon_j - \varepsilon_*$, $\mu_j - \mu_*$, $\sigma_j - \sigma_* \in C_0^7(\Omega)$, $j = 1, 2$. If

$$\Lambda_{\alpha_1,\beta_1} = \Lambda_{\alpha_2,\beta_2}, \tag{0.5}$$

then there exists $\varepsilon(\Omega) > 0$ *such that*

$$(\alpha_1, \beta_1) = (\alpha_2, \beta_2) \quad \text{in } \bar{\Omega}$$

whenever

$$\|\alpha_j - \alpha_*\|_{W^{7,\infty}(\Omega)} + \|\beta_j - \beta_*\|_{W^{7,\infty}(\Omega)} < \varepsilon.$$

The general outline of the proof of Theorem 0.4 follows the same lines as the proof of the global uniqueness theorem for the inverse conductivity problem given in [S-U II]. Namely, one first proves an identity involving products of solutions of the equation under consideration. Then one constructs exponential growing solutions of the equation to obtain information, via this identity, of the Fourier transform of the unknown function. There are two main difficulties in carrying out this approach for the problem under consideration here. First, we cannot reduce Maxwell's equations to a Schrödinger-type equation. The best we can do is to reduce Maxwell's equations to a system whose principal part is the Laplacian times the identity operator. We can construct exponential growing solutions under appropiate smallness assumptions on the first-order terms. Also, in our case we have to construct global solutions in order to guarantee that the solutions constructed satisfy the condition that the electric and magnetic field be divergence-free. In order to determine the two unknowns α and β simultaneously one has to study the asymptotic expansion of these solutions in a free parameter. The second and, perhaps, the main difficulty is that such asymptotic expansions are not available in dimension 3 in general, since these solutions are global ones. We overcome this difficulty by obtaining the necessary asymptotic expansions in the directions needed. Full details are in Section 2 and 3.

After these general comments we obtain the identity that we shall use and the reduction of Maxwell's equations to a system whose principal part is the Laplacian.

Lemma 0.6. *Let* $\varepsilon_j, \mu_j > 0$, $\sigma \geqq 0$ *in* $\bar{\Omega}$ *and*

$$\alpha_j = i\omega\mu_j, \quad \beta_j = -i\omega\varepsilon_j + \sigma_j, \quad j = 1, 2.$$

Assume

$$\alpha_j - \alpha_*, \beta_j - \beta_* \in C_0^2(\Omega),$$

$$\Lambda_{\alpha_1,\beta_1} = \Lambda_{\alpha_2,\beta_2}.$$

Then

$$\int_\Omega ((\alpha_1 - \alpha_2)\, h_1 h_2 - (\beta_1 - \beta_2)\, e_1 e_2)\, dx = 0 \tag{0.7}$$

for every solution (e_j, h_j) *of*

$$\operatorname{curl} e_j = \alpha_j h_j \quad \text{in } \Omega, \quad j = 1, 2,$$

$$\operatorname{curl} h_j = \beta_j e_j \quad \text{in } \Omega, \quad j = 1, 2.$$

Proof. We have

$$\int_\Omega \alpha_1 h_1 h_2 = \int_\Omega h_2 \operatorname{curl} e_1 = \int_{\partial\Omega} e_1 \cdot (\nu \wedge h_2) - \int_\Omega e_1 \operatorname{curl} h_2.$$

Then we obtain

$$\int_\Omega \alpha_1 h_1 h_2 + \beta_2 e_1 e_2 = \int_{\partial\Omega} e_1 \Lambda_{\alpha_2,\beta_2} e_2. \tag{0.8}$$

Similarly we can prove that

$$\int_\Omega \alpha_2 h_1 h_2 + \beta_1 e_1 e_2 = \int_{\partial\Omega} e_2 \Lambda_{\alpha_1,\beta_1} e_1. \tag{0.9}$$

From (0.8) and (0.9) we conclude

$$\int_\Omega (\alpha_1 - \alpha_2)\, h_1 h_2 - (\beta_1 - \beta_2)\, e_1 e_2 = \int_{\partial\Omega} -e_2 \Lambda_{\alpha_1,\beta_1} e_1 + e_1 \Lambda_{\alpha_2,\beta_2} e_2. \tag{0.10}$$

Since $\Lambda_{\alpha,\beta}$ is selfadjoint, we deduce from (0.10) that

$$\int_\Omega (\alpha_1 - \alpha_2)\, h_1 h_2 - (\beta_1 - \beta_2)\, e_1 e_2 = \int_{\partial\Omega} e_1 (\Lambda_{\alpha_1,\beta_1} - \Lambda_{\alpha_2,\beta_2})\, e_2 = 0, \tag{0.11}$$

concluding the proof of the lemma. □

Remark 0.12. The assumptions in Lemma 0.6 can be relaxed to $\alpha_j - \alpha_* \in C^2(\bar{\Omega})$, $\beta_j - \beta_* \in C^2(\bar{\Omega})$. Also the assumption in Theorem 0.4 can be relaxed to $\alpha_j - \alpha_* \in C^7(\bar{\Omega})$, $\beta_j - \beta_* \in C^7(\bar{\Omega})$. This is done by proving that if $\alpha, \beta \in C^\infty(\bar{\Omega})$, then $\Lambda_{\alpha,\beta}$ determines $\partial^\gamma \alpha|_{\partial\Omega}$, $\partial^\gamma \beta|_{\partial\Omega}$ for all γ. This is the analog of the KOHN-VOGELIUS result for the inverse conductivity problem [K-V]. The result can be proved by using the methods of [L-U] or [S-U III], that is, by computing the full symbol of the pseudodifferential operator $\Lambda_{\alpha,\beta}$.

Lemma 0.13. *Let* $(e, h) \in C^2(\Omega)$. *Then* (e, h) *satisfies* (0.1) *if and only if it satisfies*

$$\begin{gathered} \Delta e + L_{\alpha\beta} e - \nabla \ln \alpha \cdot \nabla e + \operatorname{Hess}(\ln \beta)\, e + \alpha\beta e = 0 \quad \text{in } \Omega, \\ \operatorname{div}(\beta e) = 0 \quad \text{in } \Omega, \\ h = \frac{1}{\alpha} \operatorname{curl} e \quad \text{in } \Omega. \end{gathered} \tag{0.14}$$

or

$$\begin{gathered} \Delta h + L_{\alpha\beta} h - \nabla \ln \beta \cdot \nabla h + \operatorname{Hess}(\ln \alpha)\, h + \alpha\beta h = 0 \quad \text{in } \Omega, \\ \operatorname{div}(\alpha h) = 0 \quad \text{in } \Omega, \\ e = \frac{1}{\beta} \operatorname{curl} h \quad \text{in } \Omega. \end{gathered} \tag{0.15}$$

Here $L_{\alpha\beta}$ *is the first-order* (3×3) *system given by*

$$L_{\alpha\beta} = \left((\ln(\alpha\beta))_{x_1} \nabla,\ (\ln(\alpha\beta))_{x_2} \nabla,\ (\ln(\alpha\beta))_{x_3} \nabla\right).$$

Proof. From (0.1) we deduce that

$$-\alpha \operatorname{curl}\left(\frac{1}{\alpha} \operatorname{curl} e\right) + \alpha\beta e = 0,\ \operatorname{div}(\beta e) = 0 \quad \text{in } \Omega. \tag{0.16}$$

Thus we obtain

$$-\alpha \operatorname{curl}\left(\frac{1}{\alpha} \operatorname{curl} e\right) + \nabla\left(\frac{1}{\beta} \operatorname{div}(\beta e)\right) + \alpha\beta e = 0 \quad \text{in } \Omega. \tag{0.17}$$

By using the fact that

$$\Delta u = -\operatorname{curl}(\operatorname{curl} u) + \nabla \cdot \operatorname{div} u, \tag{0.18}$$

we get (0.14). Now if (e, h) satisfies (0.14), then

$$\operatorname{curl}\left(\frac{1}{\alpha} \operatorname{curl} e\right) = \beta e \quad \text{in } \Omega,$$

which leads to (0.1).

A completely analogous argument proves the equivalence of (0.1) and (0.15). □

In Section 1 we construct the exponential growing solutions we need. A detailed analysis of the asymptotic expansion for large frequencies of the "remainder" is necessary if we are to obtain information about both coefficients α and β when we plug these special solutions into the identity (0.7). This is done in Sections 2 and 3.

Our interest in this problem came from listening to stimulating talks by CHENEY and ISAACSON. We also thank them for making their preprint [S-I-C] available to us. While this paper was being written, we received an interesting preprint by COLTON & PÄIVÄRINTA [C-P] in which they consider the inverse scattering problem at a fixed energy for electromagnetic waves. They assume that the magnetic permeability is constant, so there is only one function, namely β, to be determined. They prove a global result in this case.

1. Special Solutions

We fix $k \in \mathbb{R}^3 - 0$. Let $\omega_1, \omega_2 \in S^2$ be such that

$$\langle k, \omega_1\rangle = \langle k, \omega_2\rangle = \langle \omega_1, \omega_2\rangle = 0 \tag{1.1}$$

where $\langle\, ,\rangle$ denotes the standard scalar product in $\mathbb{R}^3$. We also choose

$$\rho = \omega_1 + i\omega_2. \tag{1.2}$$

Let $s \in \mathbb{R}^+$. We consider

$$\xi = s\rho + i\,\frac{k}{2} + g(s)\,\omega_1, \tag{1.3}$$

$$\eta = k - \frac{i|k|^2}{2s}\,\omega_1 + \frac{g(s)}{s}\,k,$$

where

$$g(s) = \frac{|k|^2 + 4c_*}{4s + 2\sqrt{4s^2 + |k|^2 + 4c_*}} \tag{1.4}$$

and $c_* = \alpha_*\beta_*$. Notice that with the choice of ξ, η as in (1.3) we have

$$\xi^2 = c_*, \qquad \xi\cdot\eta = 0, \tag{1.5}$$

where $\cdot$ denotes the standard scalar product in $\mathbb{C}^3$. The main result of this section is the following

Theorem 1.6. *Let $\alpha = i\omega\mu$, $\beta = -i\omega\varepsilon + \sigma$ with $\mu, \varepsilon > 0$ in $\bar{\Omega}$, $\sigma \geqq 0$ in $\bar{\Omega}$. Extend $\alpha = \alpha_*$, $\beta = \beta_*$ in Ω^c. Let ξ, η be as in* (1.3). *Let $-1 < \delta < 0$. Then there exist $\tau(\Omega, \delta) > 0$, $R > 0$ such that if $s > R$ and*

$$\|\alpha - \alpha_*\|_{W^{3,\infty}(\Omega)} + \|\beta - \beta_*\|_{W^{3,\infty}(\Omega)} < \tau, \tag{1.7}$$

then there is a unique solution of (0.14) *in $\mathbb{R}^3$ of the form*

$$e = e^{x\cdot\xi}(\eta + \psi(x, \xi)), \tag{1.8}$$

with $\psi \in H^1_\delta(\mathbb{R}^3)$ and $\psi = O(1)$ as $s \to \infty$. Here $L^2_\delta(\mathbb{R}^3)$ denotes the Hilbert space

$$L^2_\delta(\mathbb{R}^3) = \{f : \textstyle\int (1 + |x|^2)^\delta\,|f(x)|^2\,dx < \infty\};$$

$H^m_\delta(\mathbb{R}^3)$ denotes the corresponding Sobolev space.

Before proceeding with the proof of Theorem 1.6 we recall a fundamental result of [S-U II].

Lemma 1.9. *Let $\zeta \in \mathbb{C}^n - \mathbb{R}^n$, $n \geqq 3$, $t \in \mathbb{C}$ with $\zeta\cdot\zeta = t$. Let $-1 < \delta < 0$. Then given $f \in H^m_{\delta+1}(\mathbb{R}^n)$, $m \geqq 0$, there exists a unique $u \in H^m_\delta(\mathbb{R}^n)$ satisfying*

$$L_\zeta u = \Delta u + 2\zeta\cdot\nabla u = f \quad \text{in } \mathbb{R}^n. \tag{1.10}$$

Moreover,

$$\|u\|_{H^m_\delta} \leqq \frac{C}{|\zeta|}\|f\|_{H^m_{\delta+1}}. \tag{1.11}$$

We denote

$$u = L_\zeta^{-1} f. \tag{1.12}$$

A consequence of Lemma 1.9 is that

$$\|L_\zeta^{-1}\|_{H^m_{\delta+1}, H^m_\delta} \leqq \frac{C}{|\zeta|}, \tag{1.13}$$

where the norm in (1.13) denotes the operator norm.

Remark 1.14. The proof in [S-U II] was given under the condition that $\zeta \in \mathbb{C}^n - 0$ with $\zeta \cdot \zeta = 0$. However, the same proof applies to this case.

An additional fact that we shall need about the solution u of (1.10) is that, under the same assumptions as in Lemma 1.9 with $f \in L^2_{\delta+1}$, u actually belongs to $H^1_\delta(\mathbb{R}^3)$. More precisely, the following lemma holds.

Lemma 1.15. *Let $-1 < \delta < 0$. Let $f \in L^2_{\delta+1}(\mathbb{R}^n)$, $n \geqq 3$. Let ζ, t, u be as in Lemma 1.9. Then*

$$u \in H^1_\delta(\mathbb{R}^n).$$

Moreover,

$$\|\nabla u\|_{L^2_\delta(\mathbb{R}^n)} \leqq C\|f\|_{L^2_{\delta+1}(\mathbb{R}^n)} \tag{1.16}$$

for some $C > 0$.

Proof. The proof of this result was given in [S II]. Since this is not readily available we give a sketch of the proof here.
Let

$$\Omega_1 = \{x \in \mathbb{R}^n,\ \tfrac{1}{2} < |x| < 3\},$$

$$\Omega_2 = \{x \in \mathbb{R}^n,\ 1 < |x| < 2\}.$$

Let $g \in L^2(\Omega_1)$. Suppose $\omega \in H^2(\Omega_1)$ is a solution of $\Delta\omega + 2l \cdot \nabla\omega = g$ where $|l| \geqq 1$, $l \in \mathbb{C}^3$. Then (see [S I])

$$\|\nabla\omega\|^2_{L^2(\Omega_2)} \leqq C(|l|^2\|\omega\|^2_{L^2(\Omega_1)} + \|f\|^2_{L^2(\Omega_1)}). \tag{1.17}$$

In our case we known already that $f \in L^2_{\text{loc}}(\mathbb{R}^n)$. Then $\omega \in H^2_{\text{loc}}(\mathbb{R}^n)$ by standard elliptic estimates. We get the result from the local estimate (1.17) by a scaling argument. Consider the transformation $T = x/R$. Let $\tilde{u}(y) = u \circ T_R^{-1}$. Then it is easy to see that $\tilde{u}$ satisfies the equation

$$\Delta\tilde{u} + 2Rl \cdot \nabla\tilde{u} = R^2\tilde{f}, \tag{1.18}$$

where $\tilde{f} = f \circ T_R^{-1}$. applying (1.17) to (1.18) in the domain Ω_1 we get

$$\|\nabla\tilde{u}\|^2_{L^2(1 \leqq |y| \leqq 2)} \leqq CR^2|l|^2\|\tilde{u}\|_{L^2(\Omega_1)} + CR^2\|\tilde{f}\|_{L^2(\Omega_1)}. \tag{1.19}$$

Therefore we conclude that since $\delta < 0$,

(1.20)
$$\int_{R \leqq |x| \leqq 2R} (1+|x|^2)^\delta |\nabla u|^2 \leqq C\left(\frac{(1+R^2)^\delta}{(1+9R^2)^\delta}+1\right) \int_{\frac{R}{2} \leqq |x| \leqq 3R} (1+|x|^2)^\delta (|l|^2 |u|^2 + |f|^2).$$

Using the fact that $(1+9R^2)/(1+R^2)$ is bounded, letting $R = 2^j$, $j = 1, 2, \ldots$, and summing over j, we obtain the desired result. □

By (1.16) we then obtain that

$$\|L_\zeta^{-1}\|_{L^2_{\delta+1}, H^1_\delta} \leqq C. \tag{1.21}$$

Proof of Theorem 1.6. The transport equation for ψ is

$$\begin{aligned} L_\xi \psi - (\nabla \ln \alpha \cdot \nabla - L_{\alpha\beta})\, \psi &+ \big(-\xi \cdot \nabla \ln \alpha \\ &+ \operatorname{Hess}(\ln \beta) + (\alpha\beta - c_*)\big)\, \psi + \big(\psi \cdot \nabla \ln(\alpha\beta)\big)\, \xi \\ &= (\xi \cdot \nabla \ln \alpha)\, \eta - \big(\eta \cdot \nabla \ln(\alpha\beta)\big)\, \xi - \big(\operatorname{Hess}(\ln \beta) + \alpha\beta - c_*\big)\, \eta. \end{aligned} \tag{1.22}$$

The left-hand side of (1.22), other than $L_\xi \psi$, involves first-order derivatives of ψ as well as a potential term. The right-hand side of (1.22) involves terms of order $O(s)$ as $s \to \infty$. Let us apply L_ξ^{-1} to both sides of (1.22). Then we must solve in $H^1_\delta(\mathbb{R}^3)$ the equation

$$(I + F_1 + F_2)\, \psi = L_\xi^{-1}(\text{right-hand side of (1.22)}), \tag{1.23}$$

where I denotes the identity operator and

$$\begin{aligned} F_1 &= L_\xi^{-1}(L_{\alpha\beta} - \nabla \ln \alpha \cdot \nabla), \\ F_2 &= L_\xi^{-1}\big(-\xi \cdot \nabla \ln \alpha + (\nabla \ln(\alpha\beta))\, \xi + \operatorname{Hess}(\ln \beta) + \alpha\beta - c_*\big). \end{aligned} \tag{1.24}$$

Now using (1.13) and (1.21) we conclude that

$$\begin{aligned} \|F_1\|_{H^1_\delta, H^1_\delta} &\leqq C \|L_\xi^{-1}\|_{L^2_{\delta+1}, H^1_\delta}\, \delta_1, \\ \|F_2\|_{H^1_\delta, H^1_\delta} &\leqq C \|L_\xi^{-1}\|_{L^1_{\delta+1}, H^1_\delta}\, \delta_2, \end{aligned} \tag{1.25}$$

where

$$\begin{aligned} \delta_1 &= \|L_{\alpha\beta} - \nabla \ln \alpha \cdot \nabla\|_{H^1_\delta, L^2_{\delta+1}}, \\ \delta_2 &= \|-\xi \cdot \nabla \ln \alpha + \big(\nabla \ln(\alpha\beta)\big)\, \xi + \operatorname{Hess}(\ln \beta) + \alpha\beta - c_*\|_{W^{1,\infty}(\Omega)}. \end{aligned} \tag{1.26}$$

Since α and β are constants outside a ball, using the estimates (1.13) and (1.21) we conclude that for ε sufficiently small in (1.7), δ_1 and δ_2 can be chosen arbitrarily small, proving the invertibility of (1.23) in $H^1_\delta(\mathbb{R}^3)$. We also observe that the estimates (1.13) and (1.21) imply that $\delta_1 = O(1)$, $\delta_2 = O(s)$ as $s \to \infty$, thereby concluding that $\psi = O(1)$ as $s \to \infty$. The last step in the proof of the theorem is to check the divergence-free condition in (0.14). The unique global solutions constructed will allow us to check this condition. We do not know how to prove it by using local solutions. From (0.17) we have

$$\beta e - \operatorname{curl}\left(\frac{1}{\alpha}\operatorname{curl} e\right) + \frac{1}{\alpha}\nabla\left(\frac{1}{\beta}\operatorname{div}(\beta e)\right) = 0. \tag{1.27}$$

Taking the divergence of (1.27) we obtain

$$\operatorname{div}(\beta e) + \operatorname{div}\left(\frac{1}{\alpha}\nabla\left(\frac{1}{\beta}\operatorname{div}(\beta e)\right)\right) = 0. \tag{1.28}$$

Let

$$p = \frac{1}{\beta}\operatorname{div}(\beta e). \tag{1.29}$$

Then (1.28) can be rewritten as

$$\operatorname{div}\left(\frac{1}{\alpha}\nabla p\right) + \beta p = 0. \tag{1.30}$$

Let us define

$$q = \frac{p}{\sqrt{\alpha}}. \tag{1.31}$$

Then q satisfies

$$\Delta q + \left(\frac{-\Delta\dfrac{1}{\sqrt{\alpha}}}{\dfrac{1}{\sqrt{\alpha}}} + \beta\alpha\right) q = 0. \tag{1.32}$$

From the construction of e as in (1.8), using the fact that $\xi\cdot\eta = 0$, we obtain

$$q = e^{x\cdot\xi}h, \tag{1.33}$$

where $h = (\xi\cdot\psi + \eta\cdot\nabla\beta + \nabla\beta\cdot\psi + \operatorname{div}\psi)\dfrac{1}{\beta\sqrt{\alpha}}$.

We find that $h \in L^2_\delta(\mathbb{R}^3)$ satisfies

$$\Delta h + 2\xi\cdot\nabla h + \left(\frac{-\Delta\dfrac{1}{\sqrt{\alpha}}}{\dfrac{1}{\sqrt{\alpha}}} + (\alpha\beta - c_*)\right) h = 0.$$

Using Lemma 1.9 we find that $h = 0$ for large ξ, implying that $\operatorname{div}(\beta e) = 0$. □

2. Asymptotics

As we mentioned in the Introduction, in order to get information about $\hat{\alpha}$ and $\hat{\beta}$ from the identity (0.7) and the special solutions (1.8) we need to compute explicitly the asymptotic expansion of ψ as $s \to \infty$. This cannot be done in general, but the special structure of Maxwell equations will allow us to do the asymptotics we need. First, we specialize ζ in Lemma 1.9.

Lemma 2.1. *Let ζ and $u(x, \zeta)$ be as in Lemma 1.9. Let $-1 < \delta, \delta' < 0$, $\delta' < \delta$. We take*

$$\zeta = s\tilde{w} \quad \text{with } s \in \mathbb{R}^+, \quad |\tilde{w}| = 1.$$

If $f \in H^1_\delta(\mathbb{R}^n)$, then

$$u(x, \zeta) = \frac{a(x, \tilde{w})}{s} + R, \tag{2.2}$$

where a is the unique solution in $L^2_{\delta'}(\mathbb{R}^n)$ to

$$2\tilde{w} \cdot \nabla a = f, \tag{2.3}$$

$$\lim_{s \to \infty} sR = 0 \quad \text{in } L^2_{\delta'}(\mathbb{R}^n).$$

Proof. Let us consider $su = v$. Then by using Lemma 1.9 we have that

$$\|v(s, x, \tilde{w})\|_{H^1_\delta} \leqq C \tag{2.4}$$

uniformly in s. Since the inclusion $H^1_\delta(\mathbb{R}^n) \hookrightarrow L^2_{\delta'}(\mathbb{R}^n)$ is compact (this in an easy consequence of Lemma 4.1 of [M]), we conclude that for every sequence $v_n = v(s_n, x, \tilde{w})$, $s_n \to \infty$, there is a convergence subsequence $v(s_{n_{(i)}}, x, \tilde{w})$. Let

$$a = \lim_{n \to \infty} v(s_{n_{(i)}}, x, \tilde{w}) \in L^2_{\delta'}(\mathbb{R}^n). \tag{2.5}$$

In principle, a depends on the sequence taken. However, we can see that this is not the case: Since u satisfies the equation

$$\Delta u + 2\zeta \cdot \nabla u = f, \qquad \zeta = s_{n_{(i)}}\tilde{w},$$

taking the limit as $n \to \infty$ of the equation we get that a as in (2.5) must satisfy

$$2\tilde{w} \cdot \nabla a = f. \tag{2.6}$$

However, there is a unique $L^2_{\delta'}(\mathbb{R}^n)$ solution of (2.6) (see the arguments in proving Corollary 3.4 in [S-U II]). Therefore

$$a = \lim_{s \to \infty} su(x, \zeta),$$

concluding the proof of the lemma. □

In our case we need to get a further term in the expansion of ψ as in (1.8). In general it is not possible to do so. To indicate under which type of assumptions this is possible, we now state a lemma whose proof we shall use later on.

Lemma 2.7. *Let δ, δ', ζ, s, $\tilde{w}$, u be as in Lemma 2.1. Assume $f \in H^2_\delta(\mathbb{R}^n)$ and $a \in W^2_0(\mathbb{R}^n)$. Then*

$$u(x, \zeta) = \frac{a(x, \tilde{w})}{s} + \frac{b(x, \tilde{w})}{s^2} + \tilde{R} \tag{2.8}$$

as $s \to \infty$, where $b \in L^2_\delta(\mathbb{R}^n)$ is the unique solution to

$$2\tilde{w} \cdot \nabla b = -\Delta a, \tag{2.9}$$

and $\lim_{s\to\infty} s^2 \tilde{R} = 0$ in $L^2_{\delta'}(\mathbb{R}^n)$.

Proof. The proof follows the same lines as Lemma 2.1. The assumption $a \in W^2_0(\mathbb{R}^n)$ is needed to use the uniqueness and existence lemma of [S-U II]; the fact that a has compact support implies $-\Delta a \in L^2_{\delta+1}$. $\square$

We now find the asymptotics of ψ that we shall need, with ψ as in (1.8). From now on we shall assume $\alpha, \beta \in C^7_0(\bar{\Omega})$. Let us set

$$\tau_1 = \|\alpha - \alpha_*\|_{C^7(\bar{\Omega})} + \|\beta - \beta_*\|_{C^7(\bar{\Omega})}. \tag{2.10}$$

Proposition 2.11. *Let ψ, τ be as in Theorem 1.6 with $\tau_1 < \tau$. Then there exist scalar functions d_1, $\tilde{d}_1$, and d_2, and vector functions D and R such that*

$$\psi = (d_1 + \tilde{d}_1)\rho + d_2 k + \frac{D}{s} + R. \tag{2.12}$$

Moreover,

$$d_1 = d_1(x, \rho, \hat{k})|k|, \quad \|d_1\|_{H^6_\delta} \leqq C\tau_1, \tag{2.13}$$

$$\tilde{d}_1 = \tilde{d}_1(x, s, \rho, k), \quad \lim_{s\to\infty} \|\tilde{d}_1\|_{H^4_{\delta'}} = 0, \tag{2.14}$$

$$d_2 = \sqrt{\frac{\alpha}{\alpha_*}} - 1, \tag{2.15}$$

(2.16)
$$D = D_0(x, \rho, \hat{k}) + D_1(x, \rho, \hat{k})|k| + D_2(x, \rho, \hat{k})\,|k|^2, \quad \|D_j\|_{H^5_\delta} \leqq C\tau_1, \quad j = 0, 1, 2,$$

$$R = R(x, s, \rho, k), \quad \lim_{s\to\infty} s\|R\|_{H^4_{\delta'}} = 0, \tag{2.17}$$

where $\hat{k} = k/|k|$, C is a positive constant independent of τ_1, and $-1 < \delta' < \delta < 0$.

Proof. We use methods similar to those used in the proof of Lemmas 2.1 and 2.7. Our assumptions on α and β imply that $\psi \in H^5_\delta(\mathbb{R}^3)$. We shall set

$$\psi = A + G,$$

where A satisfies a "transport equation" given below. As in Lemma 2.1, the function A is uniquely determined by ψ, and the remainder G satisfies

$$\lim_{s\to\infty} \|G\|_{H^4_{\delta'}} = 0 \tag{2.18}$$

for any $-1 < \delta' < \delta$. The equation determining A is

$$2\rho \cdot \nabla A - \rho \cdot \nabla \ln \alpha (k + A) + \nabla \ln(\alpha\beta) \cdot (k + A) \rho = 0. \tag{2.19}$$

Thus $A \in H^6_\delta(\mathbb{R}^3)$. We shall show that

$$A = d_1 \rho + d_2 k$$

with d_1 and d_2 as in (2.13) and (2.15). We rewrite (2.19) as

$$2\rho \cdot \nabla \left(\frac{1}{\sqrt{\alpha}} A \right) = - \frac{1}{\sqrt{\alpha}} (\nabla \ln(\alpha\beta) \cdot A) \rho + \frac{1}{\sqrt{\alpha}} (\rho \cdot \nabla \ln \alpha) k$$

$$- \frac{1}{\sqrt{\alpha}} (k \cdot \nabla \ln(\alpha\beta)) \rho,$$

and decompose $A = A_1 + A_2$ with A_j, $j = 1, 2$, satisfying

$$2\rho \cdot \nabla \left(\frac{1}{\sqrt{\alpha}} A_2 \right) = \frac{1}{\sqrt{\alpha}} (\rho \cdot \nabla \ln \alpha) k, \tag{2.20}$$

$$2\rho \cdot \nabla \left(\frac{1}{\sqrt{\alpha}} A_1 \right) = - \frac{1}{\sqrt{\alpha}} (\nabla \ln(\alpha\beta) A_1) \rho - \frac{1}{\sqrt{\alpha}} (k \cdot \nabla \ln(\alpha\beta)) \rho \tag{2.21}$$

$$- \frac{1}{\sqrt{\alpha}} (\nabla \ln(\alpha\beta) A_2) \rho.$$

It is clear that $d_2 k$ satisfies (2.20). From (2.20) and the fact that $\rho \cdot \nabla$ is an invertible operator from $H^m_\delta(\mathbb{R}^3)$ to $H^m_{\delta+1}(\mathbb{R}^3)$, we see that

$$A_2 = d_2 k. \tag{2.22}$$

Substituting (2.22) into (2.21) we get a unique solution $A_1 \in H^6_\delta(\mathbb{R})$. Since the right-hand side of (2.21) is a scalar multiple of ρ, it follows again from the invertibility of $\rho \cdot \nabla$ that there must be a scalar function d_1 satisfying (2.13), so that

$$A_1 = d_1 \rho.$$

Since $A \in H^6_\delta(\mathbb{R}^3)$, we find that $G \in H^5_\delta(\mathbb{R}^3)$. We now show that G has the decomposition given by (2.12). Substituting $\psi = d_1 \rho + d_2 k + G$ into the equation (1.22) yields

(2.23)

$$\Delta G + 2\xi \cdot \nabla G - (\xi \cdot \nabla \ln \alpha) G - \nabla \ln \alpha \cdot \nabla G + (\nabla \ln(\alpha\beta) \cdot G) sp = I_1 + I_2 + I_3,$$

where

$$I_1 = - (\Delta d_1 + ik \cdot \nabla d_1 + g\omega_1 \cdot \nabla d_1) \rho, \tag{2.24}$$

$$(2.25)\ I_2 = -(\Delta d_2 + ik\cdot\nabla d_2)\,k + \left(\frac{ik}{2}\cdot\nabla\ln\alpha\right)k + (\rho\cdot\nabla\ln\alpha)\,\frac{i|k|^2}{2}\,\omega_1$$

$$+\left(\frac{ik}{2}\cdot\nabla\ln\alpha\right)(d_1\rho + d_2k)$$

$$+\frac{i|k|^2}{2}\,(\omega_1\cdot\nabla\ln(\alpha\beta))\,\rho - \frac{i}{2}\,(k\cdot\nabla\ln(\alpha\beta))\,k$$

$$-\frac{i}{2}\,((d_1\rho + d_2k)\cdot\nabla\ln(\alpha\beta))\,k$$

$$-L_{\alpha\beta}(d_1\rho + d_2k) - \left(\mathrm{Hess}(\ln\beta) + \alpha\beta - c_*\right)(k + d_1\rho + d_2k),$$

$$(2.26)\ I_3 = -(g\omega_1\cdot\nabla d_2)\,k + (g\omega_1\cdot\nabla\ln\alpha)\,(\eta + d_1\rho + d_2k)$$

$$-\frac{i|k|^2}{2s}\left(\left(\frac{ik}{2} + g\omega_1\right)\cdot\nabla\ln\alpha\right)\omega_1 + \frac{g}{s}\,(\xi\cdot\nabla\ln\alpha)\,k$$

$$-\frac{g}{s}(k\cdot\nabla\ln(\alpha\beta))\,\xi - g(A\cdot\nabla\ln(\alpha\beta))\,\omega_1 - \frac{|k|^2}{4s}\,(\omega_1\cdot\nabla\ln(\alpha\beta))\,k$$

$$-g\left(\left(k - \frac{i|k|^2}{2s}\,\omega_1\right)\cdot\nabla\ln(\alpha\beta)\right)\omega_1$$

$$-\left(\mathrm{Hess}(\ln\beta) + \alpha\beta - c_*\right)\cdot\left(-\frac{i|k|^2}{2s}\,\omega_1 + \frac{g}{s}\,k\right)$$

$$-\left(G\cdot\nabla\ln(\alpha\beta)\right)\cdot\left(\frac{ik}{2} + g\omega_1\right) - L_{\alpha\beta}G$$

$$-\left(\mathrm{Hess}(\ln\beta) + \alpha\beta - c_*\right)G,$$

where $g = O(1/s)$ is given by (1.4). We find that $I_j, j = 1, 2, 3$, satisfies

$$(2.27)\qquad I_2 \in H^5_{\delta+1}(\mathbb{R}^3), \quad I_3 \in H^4_{\delta+1}(\mathbb{R}^3), \quad \lim_{s\to\infty}\|I_3\|_{H^4_{\delta+1}} = 0.$$

We decompose

$$(2.28)\qquad G = G_1 + G_2 + G_3,$$

where G_2 is the unique solution in $H^5_\delta(\mathbb{R}^3)$ of (2.23) with the right-hand side replaced by I_2, where G_3 is the unique solution in $H^4_\delta(\mathbb{R}^3)$ of (2.23) with the right-hand side replaced with I_3, and where $G_1 = G - G_2 - G_3$.

From (2.27) we have that

$$(2.29)\qquad \lim_{s\to\infty} s\|G_3\|_{H^4_\delta} = 0.$$

From (2.25), we have that $I_2 = O(1)$. Therefore, we can use a method similar to that which gave A to decompose

$$G_2 = \frac{D}{s} + \tilde{R}, \tag{2.30}$$

where D is the unique H^5_δ solution to

$$2\rho \cdot \nabla D - (\rho \cdot \nabla \ln \alpha) D = I_2$$

and $\lim_{s\to\infty} s\|\tilde{R}\|_{H^4_{\delta'}} = 0$. One easily checks that D satisfies (2.16).

We now consider the term G_1. From (2.18), (2.29), and (2.30) we have that

$$\lim_{s\to\infty} \| G \|_{H^4_{\delta'}} = 0. \tag{2.31}$$

We claim that there exists a scalar function $\tilde{d}_1$ satisfying (2.14) such that

$$G_1 = \tilde{d}_1 \rho. \tag{2.32}$$

Indeed, G_1 satisfies

$$\begin{aligned} &\Delta G_1 + 2\xi \cdot \nabla G_1 - (\xi_1 \cdot \nabla \ln \alpha)\, G_1 - \nabla \ln \alpha \cdot \nabla G_1 \\ &\qquad = -[(\nabla \ln(\alpha\beta) \cdot G_1)\, s + (\Delta d_1 + ik \cdot \nabla d_1 + g\omega_1 \cdot \nabla d_1)]\, \rho, \end{aligned}$$

with the right-hand side a scalar mutiple of ρ. Formula (2.32) follows easily from the fact that $\Delta + 2\xi \cdot \nabla - (\xi \cdot \nabla \ln \alpha) - \nabla \ln \alpha \cdot \nabla$ is invertible.

Finally, by defining $R = \tilde{R} + G_3$ we get

$$G_1 + G_2 + G_3 = \tilde{d}_1 \rho + \frac{D}{s} + R,$$

and by (2.29), $\lim_{s\to\infty} s\|R\|_{H^4_{\delta'}} = 0$. The proof is now complete. □

Proposition 2.33. *Let ξ, ψ, τ be as in Theorem 1.6 with $\tau_1 < \tau$. Then*

$$\xi \cdot \psi = B + R' \tag{2.34}$$

with

$$\lim_{s\to\infty} \|R'\|_{H^4_{\delta'}} = 0, \qquad \delta' < \delta,$$

where

$$B = B(x, \rho, \hat{k})\, |k|\,. \tag{2.35}$$

Moreover, there exists $C > 0$ independent of τ_1, so that

$$\|B\|_{H^5_\delta} \leqq C\tau_1 . \tag{2.36}$$

Proof. Let $\tilde{\psi} = \xi \cdot \psi$. Then using (1.19) we check that $\tilde{\psi}$ satisfies

$$\begin{aligned} &\Delta \tilde{\psi} + 2\xi \cdot \nabla \tilde{\psi} - (\xi \cdot \nabla \ln \alpha)\, \tilde{\psi} - \nabla \ln \alpha \cdot \nabla \tilde{\psi} \\ &\qquad = -\xi L_{\alpha\beta} \tilde{\psi} - \xi\,(\mathrm{Hess}(\ln \beta) + \alpha\beta - c_*)\,(\eta + \tilde{\psi})\,. \end{aligned} \tag{2.37}$$

Since $\psi \in H^5_\delta(\mathbb{R}^3)$ and the right-hand side of (2.37) is of order $O(s)$ as $s \to \infty$, it follows that $\tilde{\psi}$ is in $H^5_{\delta+1}(\mathbb{R}^3)$. Therefore, there exists a unique $\tilde{\psi} \in H^5_\delta(\mathbb{R}^3)$ satisfying (2.37) and, moreover, $\tilde{\psi} = O(1)$ as $s \to \infty$. Using arguments analogous to the ones in the proof of Proposition 2.11 we obtain the expansion

$$\tilde{\psi} = B + R'$$

where B satisfies

$$2\rho \cdot \nabla B - (\rho \cdot \nabla\alpha) B = -\rho L_{\alpha\beta} A - \rho\left(\operatorname{Hess}(\ln \beta) + \alpha\beta - c_*\right)(k + A), \tag{2.38}$$

and the function R' satisfies

$$\lim_{s\to\infty} \|R'\|_{H^4_{\delta'}} = 0, \qquad \delta' < \delta. \tag{2.39}$$

From (2.13) and (2.15) it follows that the right-hand side of (2.38) belongs to $H^5_{\delta+1}(\mathbb{R}^3)$; hence $B \in H^5_\delta(\mathbb{R}^3)$. It is easy to check that B satisfies (2.35) and (2.36). □

3. Proof of Theorem 0.4

We use the identity (0.7) and the special solutions (1.8). We take ρ as in (1.2), and we choose ξ_1, η_1 as in (1.3) and ξ_2, η_2 as in (1.3) with ρ replaced by $-\rho$. Our solution e_j is given by

$$e_j = e^{x\cdot\xi_j}(\eta_j + \psi_j), \quad j = 1, 2. \tag{3.1}$$

Then we have

$$e_1 e_2 = e^{ix\cdot k}(\eta_1\eta_2 + \eta_1\psi_2 + \eta_2\psi_1 + \psi_1\psi_2). \tag{3.2}$$

Using (1.3) and (2.11) we get

$$e_1 e_2 = e^{ix\cdot k}\left(|k|^2 + k\cdot A^{(1)} + k \cdot A^{(2)} + A^{(1)}\cdot A^{(2)} + O\left(\frac{1}{s}\right)\right), \tag{3.3}$$

where $A^{(j)} = d_1^{(j)}\rho + d_2^{(j)}k$. (See (2.12).)

The magnetic field h_j involves derivatives of the electric field, and therefore we must be careful in determining the hehavior of $h_1 h_2$ as $s \to \infty$, as well as its dependence on k. Using (0.10), we have

$$\begin{aligned} h_j &= \frac{1}{\alpha_j}\operatorname{curl}\left(e^{x\cdot\xi_j}(\eta_j + \psi_j)\right) \\ &= e^{x\cdot\xi_j}\left(\xi_j \times (\eta_j + \psi_j) + \operatorname{curl}\psi_j\right), \quad j = 1, 2. \end{aligned} \tag{3.4}$$

Now by a direct computation

$$(\xi_1 \times \eta_1)\cdot(\xi_2\times\eta_2) = \tfrac{1}{2}|k|^4 - |k|^2 c_* + O\left(\frac{1}{s}\right). \tag{3.5}$$

Therefore

$$\alpha_1\alpha_2 h_1 h_2 = e^{ix\cdot k}\left(\tfrac{1}{2}|k|^4 - |k|^2 c_* + \sum_{j=1}^{4} I_j\right) + O\left(\frac{1}{s}\right), \tag{3.6}$$

where

$$\begin{aligned} I_1 &= \operatorname{curl}\psi_1\cdot\operatorname{curl}\psi_2,\\ I_2 &= (\xi_1\times\psi_1)\cdot(\xi_2\times\psi_2),\\ I_3 &= (\xi_1\times\psi_1)\cdot(\xi_2\times\eta_2) + (\xi_2\times\psi_2)\cdot(\xi_1\times\eta_1),\\ I_4 &= (\xi_1\times(\eta_1+\psi_1))\cdot\operatorname{curl}\psi_2 + (\xi_2\times(\eta_2+\psi_2))\cdot\operatorname{curl}\psi_1. \end{aligned} \tag{3.7}$$

The strategy of our proof is to find an asymptotic expansion in s of the left-hand side of (0.7) after substituting (3.3) and (3.6). By (3.3), the term $e_1 e_2 = O(1)$ as $s\to\infty$. We show below that the term given by (3.6) is $O(s)$. We then determine in the asymptotic expansion the coefficients of the terms of order $O(s)$ and $O(1)$. (It turns out that the higher-order terms do not give any useful information.) Then the identy (0.7) implies that these coefficients must be zero. This leads to an integral equation in the Fourier transform space for $\left(\frac{1}{\alpha_1}-\frac{1}{\alpha_2}\right)$ and $\left(\frac{1}{\beta_1}-\frac{1}{\beta_2}\right)$. To prove the uniqueness result on Theorem (0.4) it will be crucial to known the dependence on k of the coefficients in the asymptotic expansion.

In the first step of the proof we determine the asymptotic expansion in s of the terms I_j, $j=1,2,3,4$, as well as their dependence on k. From (2.12) we have

$$I_1 = \operatorname{curl}(d_1^{(1)}\rho + d_2^{(1)}k)\cdot\operatorname{curl}(-d_1^{(2)}\rho + d_2^{(2)}k) + o(1). \tag{3.8}$$

Thus,

$$I_1 = I_1(x,\rho,\hat{k})|k|^2 + o(1), \qquad \|I_1\|_{W^{5,1}(\Omega)} \leqq C\tau_1. \tag{3.9}$$

The term I_2 can be written as

$$\begin{aligned} I_2 &= (\xi_1\times\psi_1)\cdot(\xi_2\times\psi_2)\\ &= (\xi_1\cdot\xi_2)(\psi_1\cdot\psi_2) - (\xi_1\cdot\psi_2)(\xi_2\cdot\psi_1)\\ &= (\xi_1\cdot\xi_2)(\psi_1\cdot\psi_2) + (k\cdot\psi_1)(k\cdot\psi_2)\\ &\quad - (\xi_2\cdot\psi_2)(\xi_1\cdot\psi_1) + (\xi_2\cdot\psi_2)(ik\cdot\psi_1) + (ik\cdot\psi_2)(\xi_1\cdot\psi_1). \end{aligned} \tag{3.10}$$

Using (1.3) we conclude that

$$\xi_1\cdot\xi_2 = -\tfrac{1}{2}|k|^2 - c_* + O\left(\frac{1}{s}\right), \tag{3.11}$$

and from (2.12)

$$\psi_1\cdot\psi_2 = |k|^2\left(\sqrt{\frac{\alpha_1}{\alpha_*}}-1\right)\left(\sqrt{\frac{\alpha_2}{\alpha_*}}-1\right) + o(1), \tag{3.12}$$

$$k\cdot\psi_j = |k|^2\left(\sqrt{\frac{\alpha_j}{\alpha_*}}-1\right) + o(1), \quad j=1,2. \tag{3.13}$$

Now using (2.34), (2.35) as well as (3.10)–(3.13) we obtain

$$I_2 = \frac{|k|^4}{2}\left(\sqrt{\frac{\alpha_1}{\alpha_*}}-1\right)\left(\sqrt{\frac{\alpha_2}{\alpha_*}}-1\right) + I_{21}(x,\rho,\hat{k})|k|^2 + I_{22}(x,\rho,\hat{k})|k|^3 + o(1), \tag{3.14}$$

where I_{21}, I_{22} satisfy

$$\|I_{2j}\|_{W^{s,1}(\Omega)} \leqq C\tau_1, \quad j=1,2. \tag{3.15}$$

The term I_3 can be written as

$$\begin{aligned}-I_3 &= (\xi_2\cdot\psi_1)(\xi_1\cdot\eta_2) - (\xi_1\cdot\xi_2)(\psi_1\cdot\eta_2)\\ &\quad + (\xi_1\cdot\psi_2)(\xi_2\cdot\eta_1) - (\xi_1\cdot\xi_2)(\psi_2\cdot\eta_1)\\ &= -(\xi_1\cdot\psi_1)(\xi_1\cdot\eta_2) + i(k\cdot\psi_1)(\xi_1\cdot\eta_2) - (\xi_1\cdot\xi_2)(\psi_1\cdot\eta_2)\\ &\quad - (\xi_2\cdot\psi_2)(\xi_2\cdot\eta_1) + i(k\cdot\psi_2)(\xi_2\cdot\eta_1) - (\xi_1\cdot\xi_2)(\psi_2\cdot\eta_1).\end{aligned}$$

By using (2.12), (3.11) and noting that $\eta_j = k + O(1/s)$, $j=1,2$, we conclude that

$$\begin{aligned}&i(k\cdot\psi_1)(\xi_1\cdot\eta_2) - (\xi_1\cdot\xi_2)(\psi_1\cdot\eta_2) + i(k\cdot\psi_2)(\xi_2\cdot\eta_1) - (\xi_1\cdot\xi_2)(\psi_2\cdot\eta_1)\\ &\qquad = c_*|k|^2\left[\left(\sqrt{\frac{\alpha_1}{\alpha_*}}-1\right)+\left(\sqrt{\frac{\alpha_2}{\alpha_*}}-1\right)\right] + O\left(\frac{1}{s}\right).\end{aligned}$$

Therefore, by (2.34), we get

$$I_3 = I_{31}(x,\rho,\hat{k})|k|^2 + I_{32}(x,\rho,\hat{k})|k|^3 + O\left(\frac{1}{s}\right), \tag{3.16}$$

with

$$\|I_{3j}\|_{W^{s,1}(\Omega)} \leqq C\tau_1, \quad j=1,2. \tag{3.17}$$

However, the term I_4 is $O(s)$. We use the decomposition (2.12). The first term of I_4, as in (3.7), can be written as $II_1 + II_2 + II_3 + O(1/s)$, where

$$\begin{aligned} II_1 &= \big(s\rho\times(k+\psi_1)\big)\cdot \operatorname{curl}\psi_2,\\ II_2 &= \big(\tfrac{1}{2}\, ik\times(k+\psi_1)\big)\cdot\operatorname{curl}\psi_2,\\ II_3 &= \big(\rho\times(-\tfrac{1}{2}\,i\,|k|^2\,\omega_1+\psi_1)\big)\cdot\operatorname{curl}\psi_2.\end{aligned} \tag{3.18}$$

Using (2.12) and noting that $\rho\times k = i\,|k|\,\rho$, we have

$$\operatorname{curl}\psi_2 = (\nabla d_1^{(2)}\times -\rho) + (\nabla\tilde{d}_2^{(2)}\times -\rho) + (\nabla d_2^{(2)}\times k) + \frac{1}{s}\operatorname{curl} D^{(2)} + o\left(\frac{1}{s}\right),$$

$$s\rho\times\psi_1 = i\,|k|\;s\,d_2^{(1)}\rho + (\rho\times D^{(1)}) + o(1).$$

Thus,

$$\begin{aligned}(3.19)\qquad \Pi_1 &= i|k|\, s\rho\cdot(\nabla d_2^{(2)}\times k) + i|k|\,\rho\operatorname{curl} D^{(2)} + o(1)\\ &\quad + i|k|\, s\, d_2^{(1)}\rho\cdot(\nabla d_2^{(2)}\times k) + i|k|\, d_2^{(1)}\rho\cdot\operatorname{curl} D^{(2)} + o(1).\end{aligned}$$

Formula (3.19) gives the asymptotic expansion of Π_1 in s up to order $o(1)$.

From (3.18), it is easy to see that Π_2 and Π_3 are $O(1)$. Using arguments similar to those used to analyze the terms I_j, $j = 1, 2, 3$, we obtain

$$(3.20)\qquad \Pi_2 = \Pi_2(x, \rho, \hat{k})|k|^3 + o(1),$$

$$(3.21)\qquad \Pi_3 = \Pi_{31}(x, \rho, \hat{k})|k|^2 + \Pi_{32}(x, \rho, \hat{k})|k|^3 + o(1),$$

where

$$(3.22)\qquad \|\Pi_2\|_{W^{4,1}(\Omega)} + \|\Pi_{31}\|_{W^{4,1}(\Omega)} + \|\Pi_{32}\|_{W^{4,1}(\Omega)} \leqq C\tau_1.$$

In a completely analogous fashion we analyze the second term of I_4 to obtain expansions and estimates similar to (3.20), (3.21), and (3.22).

Now from (3.9), (3.14), (3.16), (3.19), (3.20), and (3.21) we obtain

Proposition 3.23. *Let I_j, $j = 1, 2, 3, 4$ be as in* (3.7). *Then there exist functions U, V, U_j, V_j, $j = 0, 1, 2, 3$, such that*

$$\sum_{j=1}^{4} I_j = \frac{|k|^4}{2}\left(\sqrt{\frac{\alpha_1}{\alpha_*}} - 1\right)\left(\sqrt{\frac{\alpha_2}{\alpha_*}} - 1\right) + sU(x, \rho, k) + V(x, \rho, k) + o(1),$$

where

$$V(x, \rho, k) = \sum_{j=0}^{3} V_j(x, \rho, \hat{k})|k|^j.$$

Moreover,

$$\|V_j\|_{W^{4,1}(\Omega)} \leqq C\tau_1, \quad j = 0, 1, 2, 3.$$

Remark. The function U in Proposition 3.23 can be computed explicitly. We have

$$U(x, \rho, k) = |k|^2\left(\sqrt{\frac{\alpha_1}{\alpha_*}}\,\rho\cdot\nabla\left(\sqrt{\frac{\alpha_2}{\alpha_*}}\right) - \sqrt{\frac{\alpha_2}{\alpha_*}}\,\rho\cdot\nabla\left(\sqrt{\frac{\alpha_1}{\alpha_*}}\right)\right).$$

It is not difficult to show that this term gives a contribution equal to zero in term $O(1)$ in the expansion of (0.7) in s.

Now we come to the second step in the proof of Theorem 0.4. Let us denote

$$(3.24)\qquad \gamma = \left(\sqrt{\frac{\alpha_1}{\alpha_*}} - 1\right)\left(\sqrt{\frac{\alpha_2}{\alpha_*}} - 1\right).$$

Substituting (3.3) and (3.6) into the identity (0.7), equating the coefficients of $O(1)$ and $O(s)$ to zero, and passing to the limit, we obtain two integral identities. A computation shows that the coefficient of $O(1)$ is zero and that the coefficient of $O(s)$ is given by

(3.25)
$$\int_\Omega e^{ix\cdot k}\left[\beta_*^2\left(\frac{1}{\beta_2}-\frac{1}{\beta_1}\right)|k|^2-\left(\frac{1}{\alpha_2}-\frac{1}{\alpha_1}\right)(1+\gamma)\frac{|k|^4}{2}+\left(\frac{1}{\alpha_2}-\frac{1}{\alpha_1}\right)|k|^2 c_*\right]$$
$$+\int_\Omega e^{ix\cdot k}\left[\left(\frac{1}{\beta_2}-\frac{1}{\beta_1}\right)S_{12}(x,\rho,k)+\left(\frac{1}{\alpha_2}-\frac{1}{\alpha_1}\right)S_{11}(x,\rho,k)\right]=0,$$

where

(3.26)
$$S_{12}(x,\rho,k) = ((\beta_1\cdot\beta_2)-\beta_*^2)|k|^2+(k\cdot A^{(2)}+k\cdot A^{(1)}+A^{(1)}\cdot A^{(2)})(x,\rho,k),$$
$$S_{11}(x,\rho,k) = -V(x,\rho,k),$$

with $A^{(j)}=d_1^{(j)}\zeta+d_2^{(j)}k$ as in (2.2).

At this point, we choose ρ as a function of k. From (1.1) and (1.2) we see that one can choose $\omega_1(k)$ and $\omega_2(k)$ as two mutually orthogonal tangent vector fields in S^2, so that $\omega_1(k)$ and $\omega_2(k)$ are piecewise continuous functions of k. Then $\rho(k)$ is also a piecewise continuous function of k. For simplicity we write

$$S_{1j}(x,k)=S_{1j}(x,\rho(k),k),\quad j=1,2.$$

We denote by $S_{11,j}(x,\hat{k})$ and $S_{12,j}(x,\hat{k})$, $j=0,1,2,3$, the coefficient of the jth power of $|k|$ in the k expansion of S_{11} and S_{12}, respectively. Then from (2.13), (2.15), and Proposition 3.23 we obtain the estimate

$$\|S_{11,j}\|_{W^{4,1}(\Omega)}+\|S_{12,j}\|_{W^{4,1}(\Omega)}\leqq C\varepsilon_1,\quad j=0,1,2,3. \tag{3.27}$$

Now we take solutions of (0.11) for the magnetic field h of the form (1.8), and we proceed in a completely analogous fashion. We conclude that

(3.28)
$$\int_\Omega e^{ix\cdot k}\left[\alpha_*^2\left(\frac{1}{\alpha_*}-\frac{1}{\alpha_1}\right)|k|^2-\left(\frac{1}{\beta_2}-\frac{1}{\beta_1}\right)(1+\tilde{\gamma})\frac{|k|^4}{2}+\left(\frac{1}{\beta_2}-\frac{1}{\beta_1}\right)|k|^2 c_*\right]$$
$$+\int_\Omega e^{ix\cdot k}\left[\left(\frac{1}{\alpha_2}-\frac{1}{\alpha_1}\right)S_{22}(x,k)+\left(\frac{1}{\beta_2}-\frac{1}{\beta_1}\right)S_{21}(x,k)\right]=0,$$

where

$$\tilde{\gamma}=\left(\sqrt{\frac{\beta_1}{\beta_*}}-1\right)\left(\sqrt{\frac{\beta_2}{\beta_*}}-1\right).$$

and the functions S_{21}, S_{22} satisfy

$$S_{2l}(x, k) = \sum_{j=0}^{3} S_{2l,j}(x, \hat{k})|k|^j, \quad l = 1, 2, \tag{3.29}$$

with

$$\|S_{21,j}\|_{W^{4,1}(\Omega)} + \|S_{22,j}\|_{W^{4,1}(\Omega)} \leqq C\tau_1, \quad j = 0, 1, 2, 3. \tag{3.30}$$

We also denote

$$U = \frac{1}{\alpha_2} - \frac{1}{\alpha_1}, \quad V = \frac{1}{\beta_2} - \frac{1}{\beta_1}. \tag{3.31}$$

We now state the main result in this second step.

Proposition 3.32. *Let U and V be as in* (3.31). *Then, for τ_1 small enough,*

$$\|U\|_{H^1(\Omega)} + \|V\|_{H^1(\Omega)} \leqq C(\|U\|_{L^2(\Omega)} + \|V\|_{L^2(\Omega)}), \tag{3.33}$$

where C depends only on $\|\alpha_j\|_{W^{7,\infty}(\Omega)}$ and $\|\beta_j\|_{W^{7,\infty}(\Omega)}$, $j = 1, 2$.

Before giving a proof of this result, we define the operator $\mathscr{S}_{ij,l}$, $1 \leqq i$, $j \leqq 2$, $l = 0, 1, 2, 3$, by

$$\mathscr{S}_{ij,l}(f)(k) = \int e^{ix\cdot k} S_{ij,l}(x, k)\chi(x) f(x)\, dx, \tag{3.34}$$

where χ is a cut-off function in $C_0^\infty(\mathbb{R}^3)$ satisfying $\chi = 1$ on $\operatorname{supp}\alpha \cup \operatorname{supp}\beta$. We claim that $\mathscr{S}_{ij,l}$ is a bounded map from $L^2(\Omega)$ to $L^2(\mathbb{R}^3)$. This is proved by using a slight variation in the proof of Theorem 18.1.11′ in [H], together with the estimates (3.27) and (3.30). We include here a short proof of this result for the sake of completeness.

We have

$$\mathscr{S}_{ij,l}(f)(k) = \int_{\mathbb{R}^3} \hat{S}^*_{ij,l}(\eta - k, k)\hat{f}(\eta)\, d\eta,$$

where ^ denotes the Fourier transform in x and

$$\mathscr{S}^*_{ij,l} = \mathscr{S}_{ij,l}\chi.$$

From (3.27) and (3.30) we readily see that

$$(1 + |\eta|)^4 |\hat{\mathscr{S}}^*_{ij,l}(\eta - k, k)| \leqq CM \quad \forall k \in \mathbb{R}^3,$$

where

$$M = \sup_{k\in\mathbb{R}^3} \sum_{|\alpha|\leqq 4} \int_\Omega |D_x^\alpha \mathscr{S}^*_{ij,l}(x, k)|\, dx < +\infty. \tag{3.35}$$

Now using Lemma 18.1.12 in [H] we conclude that $\mathscr{S}_{ij,l}$ maps $L^2(\Omega)$ to $L^2(\mathbb{R}^3)$, with norm bounded by CM.

Proof of Proposition 3.32. Multiplying both sides of (3.35) by $1/|k|^3$ we get, for $k \neq 0$,

$$\tfrac{1}{2}|k| \ (\widehat{\hat{U}(1+\gamma)})\,(k) = -\frac{\beta_*^2}{|k|}\hat{V}(k) - \sum_{j=0}^{3} |k|^{j-3}(\mathscr{S}_{12,j}V)\,(k) - \sum_{j=0}^{3} |k|^{j-3}(\mathscr{S}_{11,j}U)\,(k)\big).$$

Thus,

$$\begin{aligned}\||k|\ (\widehat{\hat{U}(1+\gamma)})\|_{L^2(|k|>1)} &\leqq \beta_*^2\|\hat{V}\|_{L^2(|k|>1)} \\ &\quad + \sum_{j=0}^{3} (\|\mathscr{S}_{12,j}V\|_{L^2(|k|>1)} + \|\mathscr{S}_{11,j}U\|_{L^2(|k|>1)}) \\ &\leqq C(\|U\|_{L^2(\Omega)} + \|V\|_{L^2(\Omega)});\end{aligned}$$

consequently,

$$\|U(1+\gamma)\|_{H^1(\Omega)} \leqq C(\|U\|_{L^2(\Omega)} + \|V\|_{L^2(\Omega)}).$$

If γ is small, we get the same estimate for U. Similar arguments can be applied to (3.28) to yield a similar estimate for V. □

For the final step of proof of Theorem 0.4 we use the the method of contradiction. Suppose that the map $(\alpha, \beta) \to \Lambda_{\alpha,\beta}$ is not injective near a constant pair (α_*, β_*) in the C^7 topology. Then there exist two sequences of pairs: $(\alpha_1^{(n)}, \beta_1^{(n)})$, $(\alpha_2^{(n)}, \beta_2^{(n)}) \in C_0^7(\Omega)$ such that

$$(\alpha_1^{(n)}, \beta_1^{(n)}) \neq (\alpha_2^{(n)}, \beta_2^{(n)}) \quad \forall n, \tag{3.36}$$

with

$$\Lambda_{\alpha_1^{(n)},\beta_1^{(n)}} = \Lambda_{\alpha_2^{(n)},\beta_2^{(n)}} \quad \forall n, \tag{3.37}$$

$$(\alpha_j^{(n)}, \beta_j^{(n)}) \overset{C^7}{\to} (\alpha_*, \beta_*) \quad \text{as } n \to \infty. \tag{3.38}$$

We define

$$\alpha_i^{(n)}(x) = \alpha_*, \quad \beta_j^{(n)}(x) = \beta_*, \quad \forall x \in \Omega^c, \ \forall n, \ j = 1, 2. \tag{3.39}$$

Let us denote

$$U^{(n)} = \left(\frac{1}{\alpha_2^{(n)}} - \frac{1}{\alpha_1^{(n)}}\right), \quad V^{(n)} = \left(\frac{1}{\beta_2^{(n)}} - \frac{1}{\beta_1^{(n)}}\right),$$

$$\tilde{U}^{(n)} = \frac{U^{(n)}}{\sqrt{\|U^{(n)}\|^2_{L^2(\Omega)} + \|V^{(n)}\|^2_{L^2(\Omega)}}}, \quad \tilde{V}^{(n)} = \frac{V^{(n)}}{\sqrt{\|U^{(n)}\|^2_{L^2(\Omega)} + \|V^{(n)}\|^2_{L^2(\Omega)}}}.$$

Then

$$\|\tilde{U}^{(n)}\|^2_{L^2(\Omega)} + \|\tilde{V}^{(n)}\|^2_{L^2(\Omega)} = 1 \quad \forall n. \tag{3.40}$$

Since (3.25), (3.28) hold whenever $\Lambda_{\alpha_1,\beta_1} = \Lambda_{\alpha_2,\beta_2}$, we conclude that there exist functions $\mathscr{S}_{ij}^{(n)}(x,k)$, $1 \leqq i, j \leqq 2$, so that

$$\|\mathscr{S}_{ij}^{(n)}\|_{W^{4,1}(\Omega)} \leqq C\tau_n \tag{3.41}$$

with $\tau_n = \max_{j=1,2}(\|\alpha_j^{(n)} - \alpha_*\|_{W^{7,\infty}} + \|\beta_j^{(n)} - \beta_*\|_{W^{7,\infty}})$. Moreover,

$$\int_\Omega e^{ix\cdot k}[\beta_*^2 \tilde{V}^{(n)}|k|^2 - \tilde{U}^{(n)}(1+\gamma_*^{(n)})\tfrac{1}{2}|k|^4 + \tilde{U}^{(n)}|k|^2 c_*] + \int_\Omega e^{ix\cdot k}[\tilde{V}^{(n)}\mathscr{S}_{12}^{(n)}(x,k) + \tilde{U}^{(n)}\mathscr{S}_{11}^{(n)}(x,k)] = 0, \tag{3.42}$$

$$\int_\Omega e^{ix\cdot k}[\alpha_*^2|k|^2 \tilde{U}^{(n)} - \tilde{V}^{(n)}(1+\tilde{\gamma}^{(n)})\tfrac{1}{2}|k|^4 - \tilde{V}^{(n)}|k|^2 c_*] + \int_\Omega e^{ix\cdot k}[\tilde{U}^{(n)}\mathscr{S}_{22}^{(n)}(x,k) + \tilde{V}^{(n)}\mathscr{S}_{21}^{(n)}(x,k)] = 0, \tag{3.43}$$

where

$$\gamma^{(n)} = \left(\sqrt{\frac{\alpha_1^{(n)}}{\alpha_*}} - 1\right)\left(\sqrt{\frac{\alpha_2^{(n)}}{\alpha_*}} - 1\right),$$

$$\tilde{\gamma}^{(n)} = \left(\sqrt{\frac{\beta_1^{(n)}}{\beta_*}} - 1\right)\left(\sqrt{\frac{\beta_2^{(n)}}{\beta_*}} - 1\right).$$

Proposition 3.32 implies that

$$\|\tilde{U}^{(n)}\|_{H^1(\Omega)} + \|\tilde{V}^{(n)}\|_{H^1(\Omega)} \leqq C,$$

for some constant C independent of n. Thus, by applying Sobolev's embedding theorem and by extracting a subsequence, we may assume that $(\tilde{U}^{(n)}, \tilde{V}^{(n)})$ converges to a function pair (U^*, V^*) in $L^2(\tilde{\Omega})$, where $\tilde{\Omega} \supset \Omega$ is a bounded domain. From (3.40) it is clear that

$$\|U^*\|^2_{L^2(\Omega)} + \|V^*\|^2_{L^2(\Omega)} = 1, \tag{3.44}$$

$$U^*(x) = V^*(x) = 0 \quad \forall x \in \tilde{\Omega}\setminus\Omega. \tag{3.45}$$

Passing to the limits in (3.42) and (3.43) and noting, from (3.41), that

$$\gamma^{(n)}, \tilde{\gamma}^{(n)}, \mathscr{S}_{ij}^{(n)}(x,k) \xrightarrow{L^\infty(\Omega)} 0 \quad \forall k,$$

we get

$$\int_{\tilde{\Omega}} e^{ix\cdot k}(\beta_*^2 V^*|k|^2 - \tfrac{1}{2}U^*|k|^4 + U^*|k|^2 c_*) = 0, \tag{3.46}$$

$$\int_{\tilde{\Omega}} e^{ix\cdot k}(\alpha_*^2 U^*|k|^2 - \tfrac{1}{2}V^*|k|^4 + V^*|k|^2 c_*) = 0. \tag{3.47}$$

Multiplying by $|k|^{-2}$ and taking the inverse Fourier transform on the left-hand side of (3.46) and (3.47), we conclude that (U^*, V^*) is the L^2 solution of the 6×6 system

$$(3.48)\qquad \begin{aligned} (\Delta + c_*)\, U^* + \beta_*^2 V^* &= 0, \\ (\Delta + c_*)\, V^* + \alpha_*^2 U^* &= 0. \end{aligned}$$

The function (U^*, V^*) has compact support. By standard elliptic regularity we see that it is in $C_0^\infty(\tilde{\Omega})$. Now, we apply $(\Delta + c_*)$ to the first equation of (3.48). Using the second one we get

$$(\Delta + c_*)^2\, U^* - \alpha_*^2 \beta_*^2 U^* = 0.$$

Recall that $c_*^2 = \alpha_*^2 \beta_*^2$. Then we have proved that

$$\Delta^2 U^* + 2c_*\, \Delta U^* = 0.$$

By unique continuation we conclude that $U^* \equiv 0$ in $\tilde{\Omega}$. Proceeding in a completely analogous fashion we conclude also that $V^* \equiv 0$ in $\tilde{\Omega}$. However, this contradicts (3.44). The proof is now complete. □

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